Analysis and Structured Representation of the Theory of Abstract Cell Complexes Applied to Digital Topology and Digital Geometry

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Contents

1 Introduction

This paper is intended to give an overview of the theory of abstract cell complexes from the viewpoint of digital image processing. We focus on digital topology and digital geometry and try to analyse the theory in a wide variety.

In the next four sections we present the base notions and some important theorems predominantly based on the publications of V.A.Kovalevsky, but also of other authors.

Section 6 contains the structured representation of the notions and theorems, which should be understood as a map of the theory.

Beside this paper, the definitions and theorems can be watched as an HTML-presentation at the following address:

`http://www.inf.tu-dresden.de/~hs24/ACC`

2 Notions and Theorems in Classical and Digital Topology

2.1 Classical Topology

In the first section we present some elementary definitions from classical topology [14] which are necessary in the field of abstract cell complexes.

Definition 1: A *topological space* R is a pair (E, SY) consisting of a set E of abstract elements and a system $SY = \{S_1, S_2, ..., S_i, ...\}$ of subsets S_i of E. These subsets are called the *open subsets* of the space and must satisfy the following axioms:

(A1) The empty subset $\emptyset$ and the set E belong to SY.

(A2) For every family F of subsets S_i belonging to SY the union of all subsets which are elements of F must also belong to SY.

(A3) If some subsets S_1 and S_2 belong to SY then the intersection $S_1 \cap S_2$ must also belong to SY.

A topological space is called *finite* iff the set E contains a finite number of elements. A topological space is called *locally finite* if every element of E bounds a finite number of other elements.

A topological space fulfills the T_i*-separation property* and is then called T_i*-space* iff it fulfills the axiom (T_i) $(i = 0, 1, 2)$. In the field of cellular complexes we are only interested in the T_0-separation property. In the following the separation axioms (see [14], p.118) are listed for completeness:

(T_0) For any two points there is an open set containing the one but not the other point. (Kolmogoroff, 1935)

(T_1) For any two points x, y there exist two open sets G, H such that $x \in G$ and $y \notin H$ and $x \notin G$ and $y \in H$. (Fréchet, 1928)

(T_2) For any two points x, y there exist two open sets G, H such that $x \in G, y \in H$ and $G \cap H = \emptyset$. (Hausdorff, 1914)

There (T_0) is the weakest separation axiom. (T_1) is a stronger demand and each T_1-space is also a T_0-space. The strongest of the three demands is (T_2) and each T_2-space is also a T_1-space. T_2-spaces are usually called *Hausdorff-spaces.*

Definition 2: A one-to-one correspondence f between a topological space R and a topological space R' is a *homeomorphism* iff the following condition is satisfied:

$$f : R \longleftrightarrow R' \text{ continuous} \Longrightarrow f^{-1} : R' \longleftrightarrow R \text{ continuous}$$

Another term for homeomorphism is *topological correspondence.* Two topological spaces are said to be *topologically equivalent* iff there exists a homeomorphic correspondence between them.

Two subsets of a topological space can be considered as identical from a topological point of view iff there exists a homeomorphism between them.

Definition 3: A property of a subset M of a topological space R is a *topological invariant* iff the same property is also valid for the set $f(M)$, for any homeomorphism f.

The dimension of a topological space is an example for a non-trivial topological invariant.

Definition 4: A *topological neighborhood* of a point p in a topological space R is any set containing an open subset of R which contains p.

2.2 Digital Topology

In the next section the most important notions from digital topology are presented. When not explicitly marked, they can be found in [5].

Definition 5: An *abstract cell complex* (ACC) $C = (E, B, dim)$ is a set E of abstract elements provided with an antisymmetric, irreflexive, and transitive binary relation $B \subset E \times E$ called the bounding relation, and with a dimension function $dim : E \longrightarrow I$ from E into the set I of non-negative integers such that $dim(e') < dim(e'')$ for all pairs $(e', e'') \in B$.

The bounding relation B is a partial order in E. The bounding relation is denoted by $e' < e''$ which means that the cell e' bounds the cell e''. Furthermore the property that any cell can only bound cells of higher dimension is emphasized by this notation:

$$e' < e'' \Longrightarrow dim(e') < dim(e'')$$

If a cell e' bounds another cell e'' then e' is called a *side* of e''. The sides of an abstract cell e'' are not parts of e''. The intersection of two distinct abstract cells, different from that of Euclidean cells, is always empty.

If the dimension $dim(e')$ of a cell e' is equal to d then e' is called *d-dimensional* cell or a *d-cell*. An ACC is called *k-dimensional* or a *k-complex* if the dimensions of all its cells are less or equal to k. Cells of dimension k, which means cells bounding no other cells, are called *base cells*.

In the field of digital image processing we normally use regular imagecarrier. There the 0-cells are called *points*, 1-cells are called *cracks*, 2-cell are called *pixels* and 3-cells are called *voxels*.

The central facts of definition 5 can be summarized in three axioms called the cell complex axioms [14]:

(C1) From $(e', e'') \in B$ and $(e'', e''') \in B$ follows $(e', e''') \in B$ (transitivity)

(C2) From $(e', e'') \in B$ follows $dim(e') < dim(e'')$ (monotony)

(C3) For each element e' there exist only a finite number of elements e'' with $(e'', e') \in B$

In addition to (C3) and leaning to the definition of topological spaces an abstract cell complex is called *locally finite* iff each of its cells bounds only a finite number of other cells.

The following theorem was first introduced and proofed in [5]:

Theorem 1: Every finite separable topological space is an abstract cellular complex according to Definition 5.

This theorem shows that there is no finite topological structure which is different from the abstract cellular complexes. Furthermore the reversal of the theorem is valid which means that every abstract cell complex is a topological space.

Here we want to introduce a corrected version of this theorem:

Theorem 1a: Every finite T_0-separable topological space is isomorphic to an abstract cellular complex.

Definition 6: A *subcomplex* $S = (E', B', dim')$ of a given ACC $C = (E, B, dim)$ is an ACC whose set E' is a subset of E and the relation B' is an intersection of B with $E' \times E'$. The dimension dim' is equal to dim for all cells of E'.

Definition 7: A subset OS of cells of a subcomplex S of an ACC C is called *open in* S if OS contains each cell of S which is bounded by a cell of OS.

Definition 8: The smallest subset of a set S which contains a given cell $c \in S$ and is open in S is called *smallest neighborhood* of c relative to S and is denoted by $SON(c, S)$.

Furthermore be $SON^*(c, S) = SON(c, S) - \{c\}$. [13]

In earlier papers the notion *smallest open neighborhood* was used, but it was incorrect, because the smallest neighborhood is open by definition. The notation SON will be used anyway.

Definition 9: A subset CS of cells of an ACC C is called *closed* if CS contains all cells of C bounding cells of CS.

There exists a duality between the notions *open* and *closed* such that a subcomplex S is open in C iff its complement $C - CS$ is closed.

Definition 10: The smallest subset of a set S which contains a given cell $c \in S$ and is closed in S is called the *closure* of c relative to S and denoted by $Cl(c, S)$.

Furthermore be $Cl^*(c, S) = Cl(c, S) - \{c\}$. [13]

Definition 11: The *closed frontier* $Fr(S, C)$ of a subcomplex S of C *relative to* C is the subcomplex of C containing all cells $c \in C$ whose smallest neighborhood $SON(c, C)$ contains both cells of S as well as cells of the complement $C - S$.

Definition 12: The frontier F of an n-dimensional subcomplex SC of an n-dimensional ACC C is called *simple* if for each cell $c \in F$ the intersection of $SON^*(c)$ with both SC and its complement $C - SC$ is non-empty and connected. [11]

A notion which is dual to the definition of the closed frontier is the following:

Definition 13: The *open frontier* $Of(S, C)$ of a subcomplex S of a complex C *relative to* C is the subcomplex of C containing all cells $c \in C$ whose closure $Cl(c, C)$ contains both cells of S as well as cells of the complement $C - S$. [8]

Definition 14: The *boundary* ∂S of an n-dimensional subcomplex S of an n-dimensional ACC C is the union of the closures of all (n-1)-cells of C each of which bounds exactly one n-cell of S. [7]

The notions *closed frontier* and *boundary* seem to be identical at first sight. In the case of a k-cell in an n-dimensional space with $k = n$ they are identical. Both frontiers are a (k-1)-dimensional sphere.
Considering a k-cell with $k < n$ one can recognize that the closed frontier is different from the boundary of the cell. The boundary is here a (k-1)-dimensional sphere as it was in the first case. The closed frontier contains in addition to the boundary also the k-cell itself, because it contains the i-cells with $k < i \leq n$ in its smallest neighborhood.

The boundary of a subcomplex of an ACC has the expected property from the continuum, i.e. the boundary of a 2-dimensional complex is 1-dimensional and has no area. The boundary of a 3-dimensional complex ist 2-dimensional and has no volume. This property was not reached with the definition of boundary pixels in 2-dimensional digital images. Furthermore one has no more to distinguish between inner and outer boundary.

The boundary definition in digital images is an instrument to solve the problem which arises with transferring the Jordan Theorem into digital spaces.

Definition 15: Two cells e' and e'' of an ACC C are called *incident* with each other in C iff one of the following cases is valid: [7]

- $(e', e'') \in B$
- $(e'', e') \in B$
- $e' = e''$

According to that the incidence relation is symmetric, reflexive and not transitive.

Definition 16: Two cells e' and e'' of an ACC C are called *connected* to each other in C iff either e' is incident with e'' or there exists a cell $c \in C$ which is connected to both e' and e''. Because of this recursive definition the connectedness relation is the transitive hull of the incidence relation. [7]

It may be easily shown that the connectedness relation is an equivalence relation. Thus it defines a partition of an ACC C into equivalence classes called the *components* of C.

An ACC C consisting of a single component is also connected.

Definition 17: A sequence of pairwise incident cells in an ACC C of the form

$$x_0^n x_1^{n-1} x_2^n ... x_{l-1}^{n-1} x_l^n$$

where x_i^n is an n-dimensional cell and x_i^{n-1} is an (n-1)-dimensional cell of C is called an *n-dimensional path* in C.

Definition 18: An n-dimensional ACC C is called *strongly connected* if any two n-dimensional cells of C may be connected by an n-dimensional path in C.

Definition 19: Two ACC's A and A' are called *B-isomorphic* to each other if there exists a one-to-one correspondence $BI : A \longrightarrow A'$ between their cells which retains the bounding relation, but not necessarily the dimension, if thus applies:

$$\forall a_1, a_2 \in A : a_1 < a_2 \Longrightarrow BI(a_1) < BI(a_2)$$

Definition 20: An n-dimensional ACC C is called *homogeneously n-dimensional* if every k-dimensional cell of C with $k < n$ is incident with an n-cell of C.

Definition 21: A homogeneously n-dimensional ACC C is called *nonbranching* if every (n-1)-cell of C bounds at most two n-cells of C. [14]

Definition 22: A *region* is an open connected subset of the space.

Definition 23: A region R of an n-dimensional ACC C is called *solid* if every cell $c \in C$ which is not in R is incident with an n-cell of the complement $C - R$. [5]

Another definition for this notion is the following:

Definition 23a: A homogeneously n-dimensional subcomplex SC of an n-dimensional ACC C is called *solid* if its complement $C - SC$ is also homogeneously n-dimensional. [11]

Investigating the properties of homogeneously n-dimensional complexes one can see that according to Definition 23a every solid region R of an ACC C is homogeneously n-dimensional. As a generalization of this proposition the next theorem follows:

Theorem 2: Every n-dimensional region S of an n-dimensional ACC C is homogeneously n-dimensional.

Proof:

> According to Definition 22 a region is an open and connected subset of the space.
> Since S is open, S contains no k-cells with $k < n$ which are not incident with an n-cell of S, otherwise S would not be open in C.
> Thus every k-cell of S is incident with an n-cell of S which is according to Definition 20 the property of being homogeneously n-dimensional.

□

Contrary to the complement of a solid region the complement of an open connected complex is in general not homogeneously n-dimensional, because it is not necessarily connected.

Since a homogeneously n-dimensional subcomplex is not necessarily connected, i.e. it consists of more than one component, the theorem can be generalized:

Theorem 3: Every open subcomplex T of an n-dimensional ACC C is homogeneously n-dimensional.

Proof:

> Since T is open, all k-cells of T with $k < n$ are incident with an n-cell of T. This is, analogous to the proof of the previous theorem, the property of being homogeneously n-dimensional.

□

2.3 Block Complexes

If the cell complexes are applied to digital image processing it is useful to consider several cells as regions. Digital images commonly consist of homogenous regions with the same label. This label is assigned to each pixel as a gray value. Based on those homogeneous regions the following notions are defined.

Definition 24: Let M denote a subdivision of an AC complex A into subsets S_i^k. The subsets with $k = 0$ are the 0-cells of A. Each subset with $k > 0$ is combinatorial homeomorphic to an open k-dimensional ball. A bounding relation BR and a dimension function Dim are defined on M in the native way. The triple $B(A) = (M, BR, Dim)$ is called *block complex of* A; the subsets S_i^k are called *k-dimensional blocks* or *k-blocks*. [10]

Hence block cells are in contrast to the cell splitting disjoint [14], every block cell of highest dimension can be declared as an open subset of $B(A)$. So a T_0-topology can be defined on $B(A)$. In other words, $B(A)$ is a T_0-space.

Definition 25: The *incidence structure* IS of a block BC of a proper block complex K is a subcomplex of K containing all blocks incident to BC except BC itself [13]:

$$IS(BC, K) = SON^*(BC, K) \cup Cl^*(BC, K)$$

Definition 26: A block complex may be described by a data structure called the *cell list*. In the 2-dimensional case it consists of one metric and three topological sublists. The topological sublists are that of 0-, 1- and 2-dimensional block cells. The metric sublist describes the coordinates of the 0-cells and of some intermediate points in the 1-cells. [5]

In the papers [10, 13] the cell list is generalized to more than two dimensions.

2.4 Skeletons

In digital image processing skeletons of segments of an image are often needed. Using the cell complexes it is also possible to extract skeletons from segments in an image.

Definition 27: The *skeleton* of a subset T of a two-dimensional image I is the smallest subset $S \subset T$ satisfying the following conditions [12]:

1) The number of components of S equals that of T.

2) The number of components of $I - S$ equals that of $I - T$.

3) Some singularities of T are kept in S.

Singularities may be defined e.g. as the "end points" in a 2D image or "borders of layers" in a 3D image etc.

Hence according to Definition 27 the skeleton has not necessarily lower dimension than the subcomplex itself, another demand to the skeleton is added:

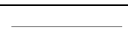
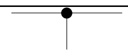
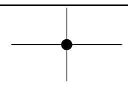
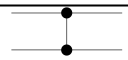
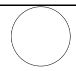
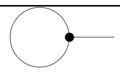
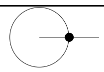
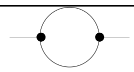
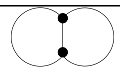
4) $dim(S) = dim(T) - 1$

This is to ensure that in a two-dimensional image the dimension of the skeletons equals 1 such that skeletons can be treated as 1-complexes.

While investigating the topological properties of two-dimensional complexes one can see that there are differences between the complexes and their skeletons. We are going to show this by the example of the images of characters.
Consider the 26 characters of the alphabet as two-dimensional images and investigate them for homeomorphisms between pairwise disjoint characters. Consider the number of components of the complement of the image of the characters as the topological invariant. The alphabet is now partitioned into three classes of characters each of which with a complement-component-number of 1, 2 or 3. This is shown in the next table.

complement-components	character
1	C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z
2	A, D, O, P, Q, R
3	B

Taking the skeletons into consideration, three different classes arise at first sight. Here the classification feature is the number of components of the two-dimensional carrier plane which are seperated by the skeleton. The difference to the two-dimensional images of the characters is that now the elements of the classes are no longer pairwise homeomorphic to each other. Hence now there are 9 classes.
In addition to the previous classification feature the elements of the skeleton-classes have the same number of line-components and the same number of branching points which are also topological invariants according to Definition 3.

components of the carrier plane	class	character
1		C, I, J, L, M, N, S, U, V, W, Z
1		E, F, G, T, Y
1		K, X
1		H
2		D, O
2		P
2		Q
2		A, R
3		B

The question now is, why reducing the dimension of a complex not only changes its spatial form but also its topological properties? One must think about the ability of the $(n-1)$-dimensional skeleton for representing topological properties of an n-complex.

Since in the example the classification of the characters is not falsified but refined, it seems to be possible to develop a character recognition on this base with eventually adding some other topological or geometrical properties.

2.5 More Notions and Problems

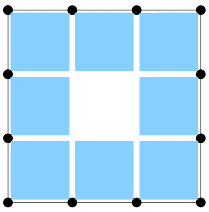
In [1] the following terms for elementary topological notions are used:

Definition28: Let C be an ACC and $A \subset C$ a subcomplex of C. Let then be

- *interior* of A: $int\ A = \bigcup_{O \subset A,\ O\ open} O$
- *exterior* of A: $ext\ A = int\ (C \setminus A)$
- *boundary* of A: $\partial A = C \setminus (int\ A \cup ext\ A)$
- *closure* of A: $\overline{A} = int\ A \cup \partial A$

With this definition there arises a contradiction to the definition of the boundary (Definition 14) which is shown with an example.

Consider first the two-dimensional pixel-array, consisting 0-, 1- and 2-cells, as the ACC C. Furthermore let $A \subset C$ be the following connected subcomplex of C.



example for a connected subcomplex $A \subset C$

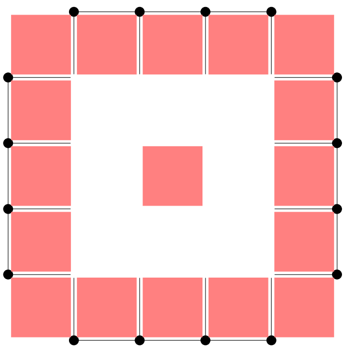
The interior of this subcomplex is according to Definition 28 the set of all 2-cells, because only the 2-cells are the open subsets of A:

$$int\ A = \{a \mid a \in A \wedge dim(a) = 2\}$$

The exterior of the subcomplex A, which is the interior of its complement relative to C, is

$$ext\ A = int\ (C \setminus A)$$

and has the following appearance:

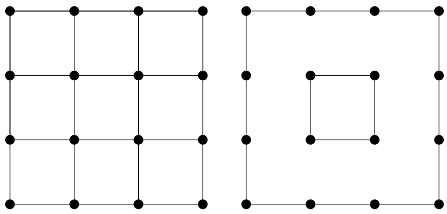


exterior of A

The boundary according to Definition 28 is

$$\partial A = C \setminus (int\ A \cup ext\ A)$$

and is now confronted to Kovalevsky's boundary for this example.



Ihle boundary (left) and Kovalevsky boundary (right)

Considering the closed frontier additionally, one can recognize that it is possible that in [1] with the notion boundary the closed frontier according to Definition 11 was meant. So it is important to use these three notions correctly, because the example shows that there are complexes where the notions have not the same results.

3 Manifolds, Balls and Spheres

3.1 Manifolds

Manifolds are elementary mathematical objects. This is the reason why some relevant notions are presented here.

An early definition for this notion, which should not be used anymore, is the following:

Definition 29: An *n-dimensional finite manifold* M_n (n-manifold) is an n-dimensional ACC satisfying the following conditions:

1) A 0-dimensional manifold M_0 consists of two 0-cells without any bounding relation.

2) An n-dimensional manifold M_n with $n > 0$ is connected.

3) For all cells $c \in M_n$ the subcomplex of all cells different from c which are incident with c is B-isomorphic to an (n-1)-dimensional manifold.

A more suitable definition which is not based on the B-isomorphism introduced by Kovalevsky is the following:

Definition 29a: An *n-dimensional combinatorial manifold* M_n without boundary is an n-dimensional ACC in which the boundary of the smallest neighborhood $SON(P, M_n)$ of each 0-cell P is homeomorhpic to an (n-1)-dimensional sphere. In a manifold with boundary the $SON(P, M_n)$ of some 0-cells P may have a boundary homeomorphic to a "half-sphere", i.e. to an (n-1)-ball. [12]

Hence a manifold is a locally strongly connected ACC.

Definition 30: A *one-dimensional quasi-manifold* is a connected one-dimensional ACC in which every 1-cell is bounded by exactly two 0-cells and every 0-cell bounds an even number (at least two) of the 1-cells. [8]

Definition 31: A *two-dimensional quasi-manifold* is a connected two-dimensional ACC in which every 2-cell is bounded by 0- and 1-cells composing a one-dimensional manifold, i.e. a cycle. Every 1-cell is bounded by exactly two 0-cells and bounds an even number (at least two) 2-cells. The 1- and 2-cells bounded by a 0-cell compose one or more subcomplexes each of which is B-isomorphic to a one-dimensional quasi-manifold. [8]

Definition 32: A strongly connected, nonbranching, homogeneously n-dimensional complex is called *pseudomanifold*. [14]

According to the definition of the pseudomanifold Rinow formulated the netxt two theorems: [14]

Theorem 4: Every homogeneously n-dimensional complex C can be represented as the union of its strong components $C = \bigcup_{i \in I} C_i$. Each strong component is a maximal, strongly connected, homogeneously n-dimensional, closed subcomplex of C and is embedded in a component of C. For two strong components C_i and C_j ($i \neq j$) the following expression holds: $dim(C_i \cap C_j) < n - 1$.

Theorem 5: Every strong component of a nonbranching complex is a pseudo-manifold.

Definition 33: The *topological genus* of the surface of a manifold is defined as the number of necessary cuts to transform the surface into a simply-connected set [2].

I.e. a manifold of genus 0 corresponds to a sphere, a manifold of genus 1 to a torus and a manifold of genus g corresponds to a sphere with g handles.

3.2 Combinatorial Topology

The field of combinatorial topology is well suited to represent objects with their topological properties with the help of computers. The theory is especially applied to digital image processing, for example to encode three-dimensional images efficiently.
The notions presented in this section are taken from [12]. More details can be found in [15].

Definition 34: A 1-cell is called *proper* if it is bounded by exactly two 0-cells.

Definition 35: An *elementary subdivision* of a proper 1-cell c^1, which is bounded by the 0-cells c_1^0 and c_2^0, replaces the complex $C' = (c_1^0 < c^1 > c_2^0)$ by the 1-complex $C'' = (c_1^0 < c_1^1 > c_3^0 < c_2^1 > c_2^0)$ with two 1-cells c_1^1 and c_2^1 and a new 0-cell c_3^0. One or both of the 0-cells c_1^0 and c_2^0 can be missing.

Definition 36: An m-complex arising through $N (N \geq 0)$ elementary subdivisions of a single proper m-cell is called *open combinatorial m-ball.* When $m = 1$ it is a sequence of pairwise incident 1- and 0-cells, starting and ending with a 1-cell. A single 1-cell is also an open combinatorial 1-ball.

Definition 37: The boundary of an open m-ball is called *combinatorial* (m-1)-*sphere.* When $m = 1$ it consists of exactly two 0-cells. The closure of an m-ball is called *closed m-ball.* The union of two closed m-balls with identical boundaries is called *combinatorial m-sphere.*

An m-cell c^m with $m > 1$ is called proper if its boundary ∂c^m is a combinatorial (m-1)-sphere.

An ACC is called proper if all its cells are proper.

An elementary subdivision in an n-complex replaces a proper m-cell c^m with $1 < m \leq n$ with two proper m-cells c_1^m, c_2^m and a new proper (m-1)-cell $c^{(m-1)}$ bounding both c_1^m and c_2^m. The boundary $\partial c^{(m-1)}$ is an (m-2)-sphere $S^{(m-2)} \subset \partial c^m$ with $\partial(c_1^m \cup c^{(m-1)} \cup c_2^m) = \partial c^m$ and $c^{(m-1)} \notin \partial c^m$. [12]

The expression $S^{(m-2)} \subset \partial c^m$ means that $S^{(m-2)}$ is a subset of the boundary which bounds the cells c_1^m and c_2^m, which is expressed by $\partial(c_1^m \cup c^{(m-1)} \cup c_2^m) = \partial c^m$. But this expression was incorrect since the boundary $\partial c^{(m-1)}$ can only belong to the boundary of the subdivided m-cell when the boundary of the m-cell is subdivided itself.

Hence the following formulation is more suitable to describe this fact:

$$S^{(m-2)} \subset \partial(c_1^m \cup c_2^m)$$

An elementary subdivision of an m-cell can be considered as a hierarchical process where the cells of dimension 1 are subdivided first, after that the cells of dimension 2 etc., up to dimension m such that the boundary of the newly added cell was added before.

Definition 38: Two proper AC complexes are called *combinatorial homeomorphic* if they possess isomorphic subdivisions.

Definition 39: In an n-dimensional space an n-cell Z^n is called a *simple cell* relative to an n-dimensional subcomplex K^n iff the intersection D of the boundaries of Z^n and K^n is homeomorphic to an (n-1)-ball.

Now one can consider the notions *closed frontier* (Definition 11), *strongly connected* (Definition 18), *homogeneously n-dimensional* (Definition 20) and *simple cell* (Definition 39).
Let B be the frontier of a four-dimensional subcomplex V of a four-dimensional ACC C where B is nonbranching. Let T be a three-dimensional subcomplex of B open in B, $Fr(T)$ its frontier relativ to B and $Z \in B$ a 3-cell. The frontier $Cl^*(Z, B)$ of Z relative to B is denoted by K.

Theorem 6: The cell Z is simple relative to T iff the intersection $D = K \cap Fr(T)$ and the complement $K - D$ are two-dimensional strongly connected complexes where D is contained in the frontier-complex of T. [10]

Theorem 7: The set $SON^*(c^k, M^n)$ of any k-cell c^k of an n-manifold M^n with $0 \leq k \leq n-1$ is B-isomorphic to an $(n-k-1)$-dimensional sphere if c^k does not belong to the boundary ∂M^n. The set $Cl^*(c^k, M^n)$ is then B-isomorphic to a $(k-1)$-dimensional sphere. [13]

Theorem 8: The boundary B of a simple and strongly connected solid subset V of the Cartesian 3D space (ACC) is a two-dimensional quasi-manifold. [8]

These three theorems have significant importance in concerning algorithms for investigate manifolds and surfaces.
In [8] an algorithm is presented (the *face code algorithm*) which analyses two-dimensional quasi-manifolds, i.e. surfaces of voxelsets. The result of the algorithm is a linear data structure.
Another application is a method to describe 3-manifolds by cell lists. See [10, 13].
Other algorithms to investigate topological properties of subsets in two- and three-dimensional digital images can be found in [12].

3.3 Interlaced Spheres

The field of interlaced spheres is theoretically and experimentally investigated in [11]. Only an important definition and a theorem will be presented here.

Definition 40: The sphere S^k is said to span the ball $B^{(k+1)}$ if $S^k = \partial B^{(k+1)}$. Two spheres S^m and S^k are called *interlaced* with each other if they do not intersect each other but each of them intersects every ball spanning the other sphere.

Theorem 9: Two spheres S^m and S^k embedded in an n-dimensional ball B^n may be interlaced with each other if and only if $m + k + 1 = n$.

4 Cartesian Complexes and Digital Geometry

4.1 Locally Finite Cartesian Spaces

Definition 41: A connected one-dimensional complex in which all cells, except two, are incident with exactly two other cells is called *topological line*. [12]

By assigning subsequent integer numbers to the cells of a topological line such a way that a cell with the number x is incident with cells having the numbers $x-1$ and $x+1$, one can define *coordinates* of a one-dimensional space. ACC's of greater dimensions are defined as Cartesian products of such one-dimensional ACC's.

Definition 42: A product ACC is called *Cartesian complex*. The one-dimensional ACC's are the *coordinate axes* A_i of the n-dimensional space. A cell of the n-dimensional Cartesian ACC C^n is an n-tupel $(a_1, a_2, ..., a_n)$ of cells of the corresponding axes: $a_i \in A_i$. [7]

The bounding relation on an n-dimensional Cartesian ACC C^n is defined as follows: An n-tupel $(a_1, a_2, ..., a_n)$ is bounding another n-tupel $(b_1, b_2, ..., b_n)$ iff for all $i = 1, 2, ..., n$ the cell a_i is incident with b_i in A_i and $dim(a_i) \leq dim(b_i)$ in A_i. The dimension of the product cell is the sum of dimensions of the factor cells in their one-dimensional spaces.

The coordinates have been introduced without having introduced either a metric, or the notion of a straight line, or the scalar product. Therefore it is correct to speak of *topological coordinates*.

For applications in digital image processing they have the disadvantage that the size of a pixel, which is the difference of the coordinate of the sides of the corresponding square, is equal to 2 rather than to 1 as it is usual in image processing. To overcome this drawback it is proposed in [7] to assign rational numbers with denominator 2 to subsequent cells of an axis. The 0-cells are assigned by fractions with an even numerator and an odd numerator is assigned to each 1-cell such that the coordinates of each 0-cell is an integer and that of 1-cells are "*half-integers*".
Hence the dimension of a product cell of an n-dimensional ACC is the number of its half-integer coordinates.

4.2 Digital Geometry

The aim in digital geometry is to determine geometric properties of an continuous object by its digital image. The main problem is the reduction of information by the digitization, so one has to approximate.

In contrast to the continuous geometry differential geometric investigations cannot be done locally. Therefore it is necessary to investigate the neighborhoods of the local points, too. For example the curvature of surfaces [1] has to be determined in this way.

To apply geometric notions in the field of abstract cell complexes it is necessary to define a metric on the topological space. The distance, which is needed to define a metric, can be based on the definition of topological coordinates introduced in the previous section. I.e. the distance between 0-cells in an n-dimensional space can be calculated by the following formula:

$$d = \left((x_{1_b} - x_{1_a})^2 + (x_{2_b} - x_{2_a})^2 + ... + (x_{n_b} - x_{n_a})^2\right)^{\frac{1}{2}}$$

Based on the metric space the following basic notions can be defined [7]:

Definition 43: A *digital half-plane* is a solid region of a two-dimensional ACC containing all pixels whose coordinates satisfy a linear inequality.

Definition 44: A non-empty intersection of digital half-planes is called *digital convex subset* of the two-dimensional space.

Definition 45: A *digital straight line segment* (DSS) is any connected subset of the boundary of a (digital) half-plane.

Definition 46: The *minimum length polygon* (MLP) is the shortest polygonal curve lying completely in the closure of the open frontier of a region. [9]

Definition 47: The *perimeter* of a region R in a two-dimensional finite space is the sum of the lengths of subsequent digital straight line segments obtained by subdividing the boundary of R into as few as possible DSS's.

As with the DSS-method the perimeter of a region can also be approximated with the MLP-method. In [9] algorithms for the MLP-method are described, experimentally investigated and compared to the DSS-method. One of the results is that both methods can also treat non-convex regions efficiently. The DSS-method provides the advantage of a faster convergence.

Definition 48: The *Minkowski-distance* is defined as follows [6]:
Let $d(p, q)$ be the Euklidean distance between the two points p and q. Let M be a set of points. Then let us call the value

$$DPS(p, M) = \min_{q \in M} d(p, q)$$

the distance from the point p to the set M. The value

$$DS(M_1, M_2) = \max_{p \in M_1} DPS(p, M_2)$$

is called the distance from the set M_1 to the set M_2. This value is obviously not symmetric. The value

$$MD(M_1, M_2) = max\ (DS(M_1, M_2), DS(M_2, M_1))$$

is called the Minkowski-distance between the sets M_1 and M_2. It is symmetric with respect to M_1 and M_2. Another term for this notion is *Hausdorff-distance*.

Kovalevsky and Fuchs introduced the following two theorems in [6]:

Theorem 10: Consider a convex subset S of the Euclidian plane and a polygon P whose Minkowski-distance to the boundary of S is less than a tolerance t. Then the perimeter of P differs from that of S by no more than $2\pi t$.

Let S be a convex Euklidean polygon whose smallest interior angle is 2β. Let $DI(S)$ be the digital image of S obtained under a digitization with a pixel size of

p. B is the crack boundary of $DI(S)$, and P a polygon approximating B in such a way that the Minkowski-distance between B and P does not exceed the value $\varepsilon \cdot p$.

Theorem 11: There exists such a pixel size p that the perimeters of S and P differ by no more than $2\pi(\frac{\sqrt{2}}{sin\beta} + \varepsilon) \cdot p$.

These two theorems have an important meaning in digital image processing, because they provide a method to approximate perimeters of regions in digital images efficiently.

Definition 49: A *digital disc* is a two-dimensional solid region containing all pixels of the plane whose coordinates satisfy the following inequality:

$$(x - x_c)^2 + (y - y_c)^2 < r^2$$

The values x and y are the half-integer coordinates of the pixels, x_c and y_c are the coordinates of the center and r is the radius of the disc. The values of x_c, y_c and r may be integer of fractional.

Definition 50: A *digital circular arc* (DCA) is any connected subset of the boundary of a digital disc.

As a generalization of the Definitions 45 and 50 the notion *digital curve* is introduced to be the boundary or a connected subset of the boundary of a two-dimensional region.

4.3 Points and Vectors

More notions from [7] are presented in the following section, because they have a practical meaning for the cell complexes applied to digital geometry.

Definition 51: A point (0-cell) C is said to be *strictly collinear* with two other points A and B if the following equality holds:

$$(x_c - x_b) \cdot (y_b - y_a) - (y_c - y_b) \cdot (x_b - x_a) = 0$$

The point C is said to lie *to the right* from the ordered pair A and B if

$$(x_c - x_b) \cdot (y_b - y_a) - (y_c - y_b) \cdot (x_b - x_a) > 0$$

and it is said to lie *to the left* from A and B if

$$(x_c - x_b) \cdot (y_b - y_a) - (y_c - y_b) \cdot (x_b - x_a) < 0$$

Definition 52: Consider all ordered pairs of points of a DSS, such that all other points of the DSS do not lie to the left of the pair. Choose the pair (A, B) with the greatest absolute difference of the coordinates $|x_b - x_a|$ or $|y_b - y_a|$. If there are points of the DSS which are strictly collinear with (A, B), choose the pair of such points which are closest to each other. Denote the points C and D. This point pair is called the *left base* of the DSS. The right base may be defined similarly.

The *slope* M/N of the base is defined by two integers:

$M = y_d - y_c$ and $N = x_d - x_c$

Definition 53: A two-dimensional vector with integer components (x, y) is called *right semi-collinear* with another integer vector (n, m) if the following inequalities hold:

$$0 \leq (x \cdot M - y \cdot N) \leq M + N - 1$$

where M and N are numerator and denominator of the irreducible fraction $M/N = m/n$. The notion of left semi-collinear vectors may be defined similarly.

Definition 54: The distance d between two points is declared *digitally equal* to a number n if the absolute difference between d and n is less or equal to the length of a pixel's diagonal ($\sqrt{2}$).

Definition 55: The *value of semi-collinearity* of a point C relative to an ordered pair of points (A, B) is declared to be 0 if C is semi-collinear with (A, B). If it is not semi-collinear, then the value of semi-collinearity is declared to be -1 or $+1$ depending on whether C lies to the left or to the right of (A, B).

We want to call the readers attention to the fact that the notions "left" and "right" are related to a coordinate system in a mathematical positive order. Hence in the field of image processing a different coordinate system is used (x-axis points to the left, y-axis points down), the roles of left and right must be changed.

Definition 56: Two figures F and G are called *congruent* with each other iff there exists such a mapping from F to G that the distance between any cells of G is digitally equal to the distance of their preimages in F and the value of semi-collinearity of any three points of G is the same as of their preimages in F.

5 Mappings among Locally Finite Spaces

In the field of mappings among locally finite spaces it is necessary to define a more general type of function [7]:

Definition 57: A *correspondence* between two locally finite spaces X and Y or a many-valued mapping of X into Y is a subset F of ordered pairs (x, y) containing all cells $x \in X$ and some cells $y \in Y$.

Definition 58: A correspondence between X and Y is called a *connectivity preserving mapping* (CPM) if the image of any connected subset of X is connected.

Definition 59: Let us denote by $V(x, y)$ the connected component of $F(x)$ containing y and by $H(x, y)$ the connected component of $F^{-1}(y)$ containig x. A correspondence F is called *simple* if for each pair $(x, y) \in F$ at most one of the sets $V(x, y)$ and $H(x, y)$ contains more that one element.

Definition 60: The *open hull* $Op(S)$ of a subset S of an ACC C is the smallest open subset of C containing S.

The notion *open hull* is a generalization of the smallest neighborhood (Definition 8). The open hull of a single cell corresponds to the smallest neighborhood of this cell.
Let $c \in C, S \subset C, S = \{c\} : Op(c) = SON(c, C)$.

Definition 61: The *closed hull* $Cl(S)$ of a subset S of an ACC C is the smallest closed subset of C containing S.

Definition 62: The *n-neighborhood* $U_n(c)$ of a cell $c \in C$ is an open subset of C satisfying the following conditions:

1) $U_0(c) = Op(c) = SON(c)$

2) $U_{n+1}(c) = Op(Cl(U_n(c)))$

Definition 63: A many-valued mapping $F : X \longrightarrow Y$ from a finite space X into a finite space Y is called *n-isomorphism* if for any two cells $x_1, x_2 \in X$ and for any cells of the images of them $y_1 \in F(x_1), y_2 \in F(x_2)$ the following two conditions are satisfied:

1) $x_2 \in U_0(x_1) \Longrightarrow y_2 \in U_n(y_1)$

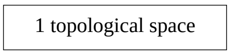
2) $x_2 \notin U_n(x_1) \Longrightarrow y_2 \notin U_0(y_1)$

6 Structured Representation

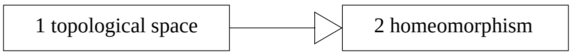
The next pages contain an illustration of the theory of abstract cell complexes which is intended to visualize the relations between the notions and theorems. The first plan is intended to show the notions used in this paper. After that partial plans are given to visualize the relations between theorems and notions in the appropriate subfield of the theory of abstract cell complexes.

The following primitives are used:

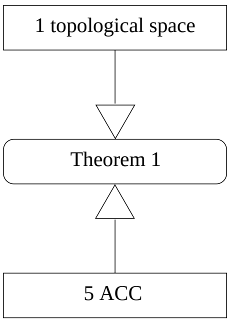
- **Notion**: We use a rectangle with the name and the number of the definition inside to illustrate a defined notion.



- **Relation**: Notions which are related to each other in any sense are connected by an arrow. The arrow is directed to the notion which is influenced by the notion where the arrow begins.

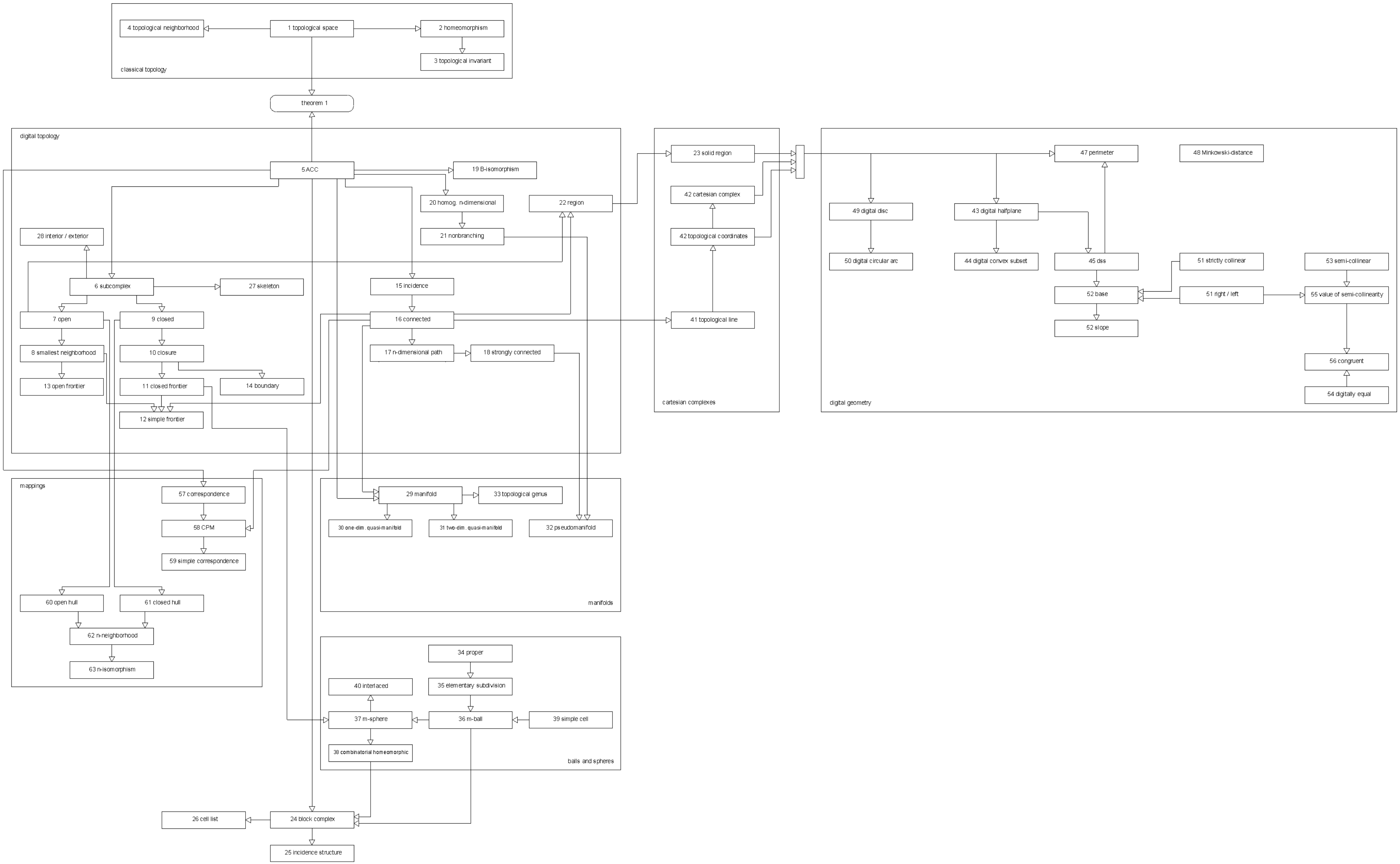


- **Theorem**: A theorem is illustrated by an rounded rectangle. The arrows pointing to the theorem come from the definitions of the notions which are used for the theorem and defined in this framework.

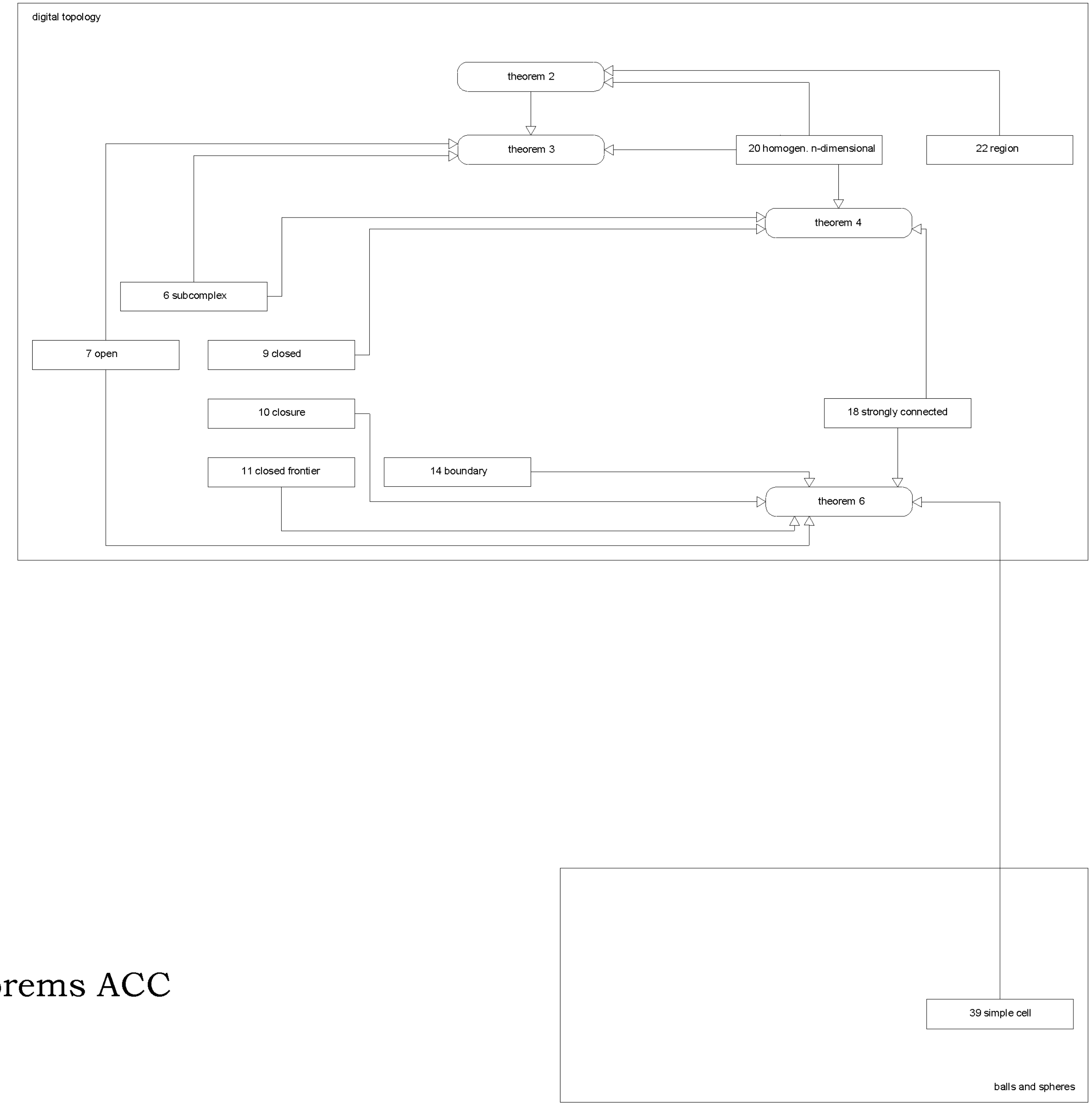


The notions in the illustration are grouped according to the theory of abstract cell complexes such that relations between the subfields can be seen. The subfields are:

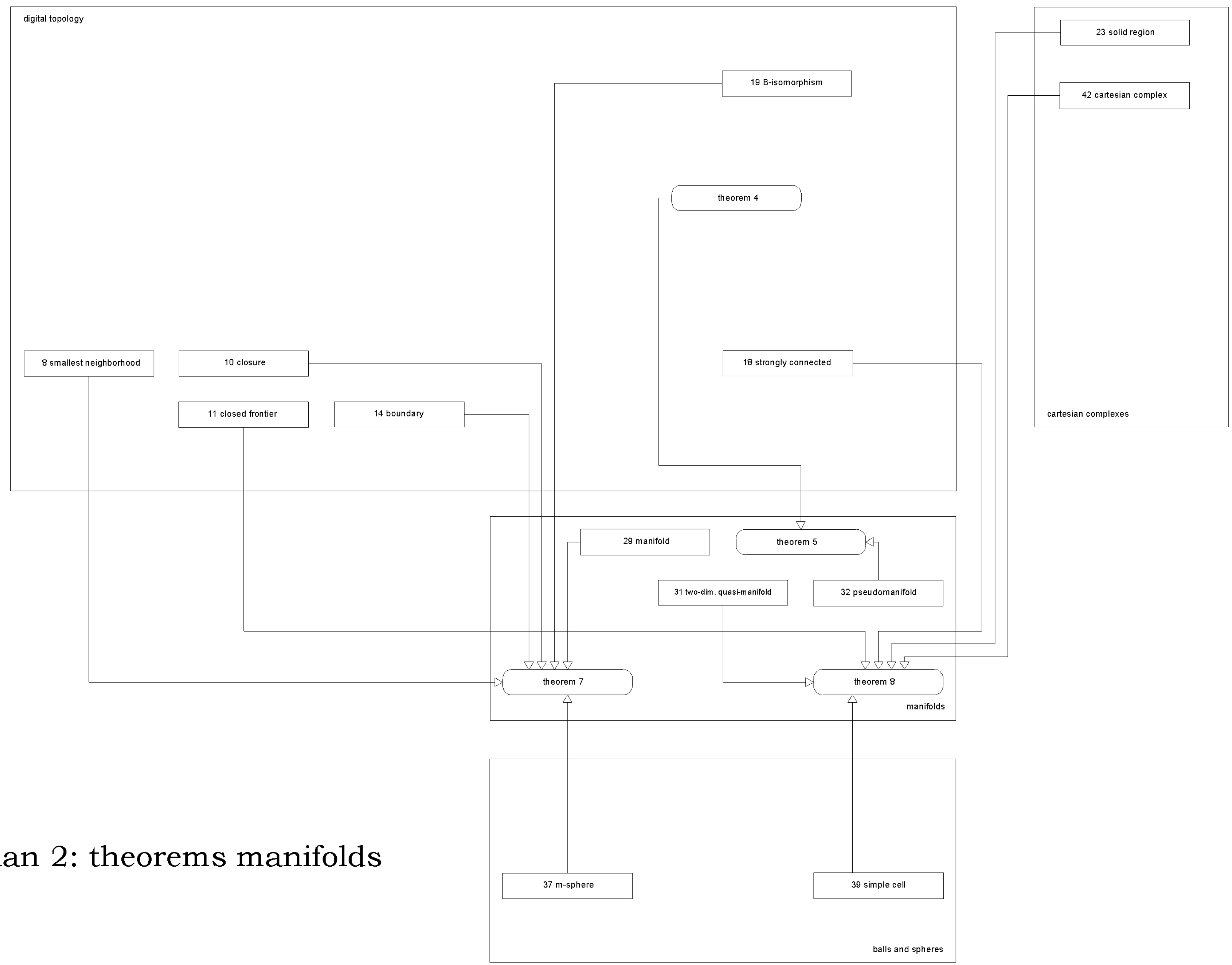
- classical topology
- digital topology
- manifolds
- combinatorial topology (balls and spheres)
- Cartesian complexes and coordinates
- digital geometry
- mappings between locally finite spaces



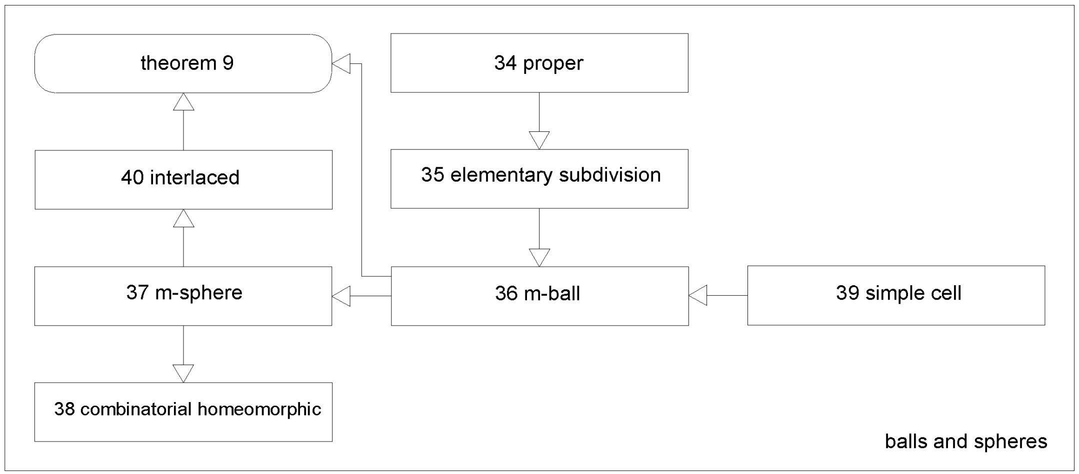
classical topology
1 topological space
2 homeomorphism
3 topological invariant
4 topological neighborhood
theorem 1
digital topology
5 ACC
6 subcomplex
7 open
8 smallest neighborhood
9 closed
10 closure
11 closed frontier
12 simple frontier
13 open frontier
14 boundary
15 incidence
16 connected
17 n-dimensional path
18 strongly connected
19 B-isomorphism
20 homog. n-dimensional
21 nonbranching
22 region
27 skeleton
28 interior / exterior
cartesian complexes
23 solid region
41 topological line
42 cartesian complex
42 topological coordinates
digital geometry
43 digital halfplane
44 digital convex subset
45 dss
47 perimeter
48 Minkowski-distance
49 digital disc
50 digital circular arc
51 strictly collinear
51 right / left
52 base
52 slope
53 semi-collinear
54 digitally equal
55 value of semi-collinearity
56 congruent
mappings
57 correspondence
58 CPM
59 simple correspondence
60 open hull
61 closed hull
62 n-neighborhood
63 n-isomorphism
manifolds
29 manifold
30 one-dim. quasi-manifold
31 two-dim. quasi-manifold
32 pseudomanifold
33 topological genus
balls and spheres
34 proper
35 elementary subdivision
36 m-ball
37 m-sphere
38 combinatorial homeomorphic
39 simple cell
40 interlaced
24 block complex
25 incidence structure
26 cell list



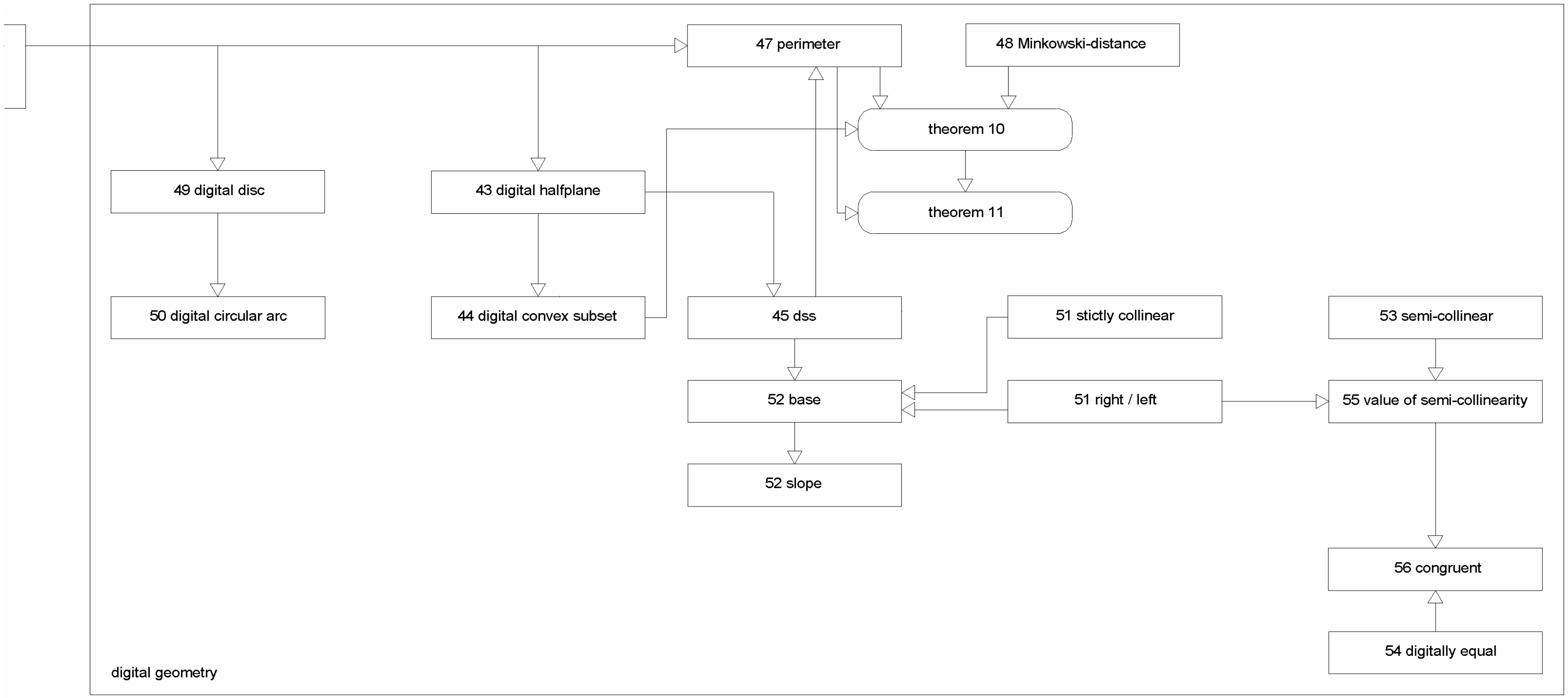
partial plan 1: theorems ACC



partial plan 2: theorems manifolds



partial plan 3: theorem interlaced spheres



partial plan 4: theorems digital geometry

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