gipsa-lab

Grenoble | images | parole | signal | automatique | laboratoire

Digital geometry: digital objects analysis

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Digital sets/objects

Digital set

Set of points in $\mathbb{Z}^n$

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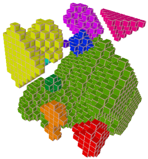
Digital set

Set of points in $\mathbb{Z}^n$

Digital object

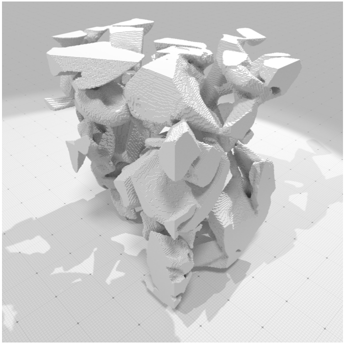
Set of points in $\mathbb{Z}^n$ + topology



What can we do from here ?

How to compute the *circularity*, *area* of these objects ?

How to compute the *curvature*, *local thickness* on this object ?



[Levallois 15, ANR digitalSnow, laboratoires LIRIS / LAMA / 3SR / MétéoFrance / CEN - CNRM GAME]

Plan

Outline

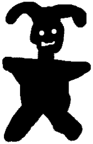
Plan

- Transformations
  - Distance Transform
  - Medial axis
  - Skeleton
- Measurements

Distance Transform

The problem

Given a digital set S, label each $p \in S$ with the distance to the closest point $q \notin S$.



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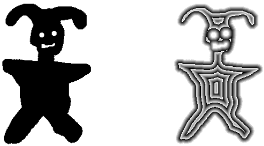
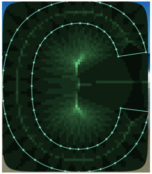


Applications

$\Rightarrow$ *signed distance field*

- Measures : thickness, differential operators on the boundary
- digital image processing : blurring effects, skeleton
- motion planning, pathfinding
- font smoothing, rendering

[http ://gamma.web.unc.edu/]



[Glyphy]

$\Rightarrow$ *distance between digital points ?*

Distance - first trial

Let p, q be two points in $\mathbb{Z}^n$.

What is the distance between p and q ?

Euclidean distance ?

$d_2(p,q) = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2}$

$\Rightarrow$ not always an integer !... and we don't like roundings...

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d_1, d_∞ distances

$d_1(p,q) = \sum_{i=1}^{n} |p_i - q_i| \qquad d_\infty(p,q) = \max_{i=1..n} |p_i - q_i|$

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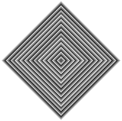
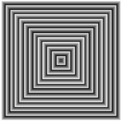




Chamfer distances

Idea [Montanari 68, Borgefors 84]

A finite set of displacements + a weight for each displacement.

Displacements

	a	
a	0	a
	a	

$a = 1 \Rightarrow d_1$

b	a	b
a	0	a
b	a	b

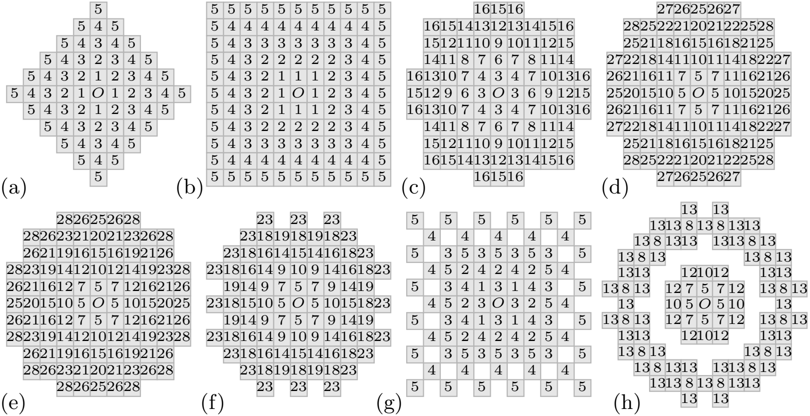
$a = b = 1 \Rightarrow d_\infty$

$2b$	c	$2a$	c	$2b$
c	b	a	b	c
$2a$	a	0	a	$2a$
c	b	a	b	c
$2b$	c	$2a$	c	$2b$

Distance

Distance between p and q = weight of the "lighter" path using only prescribed displacements.

How to set the weights ?



[Géométrie discrète et images numériques, ouvrage collectif, 2007]

Some conditions on the weights

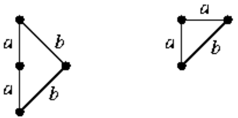
Axioms

1. $d(p,q) \geq 0, d(p,q) = 0 \Leftrightarrow p = q$ (positive, definite)
2. $d(p,q) = d(q,p)$ (symmetric)
3. $\forall r \in E,\ d(p,q) \leq d(p,r) + d(r,q)$ (triangular inequality)

Conditions

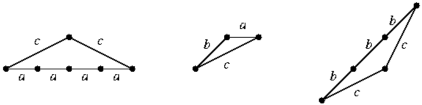
3×3

$0 < a \leq b \leq 2a$



5×5

$0 < 2a \leq c \leq a + b$
$3b \leq 2c$



Other integer distances

Other path-based distance

- neighbourhood sequences : weights and displacements vary at each step [Rosenfeld 68, Strand 07, Normand et al. 2013]

"Integer" Euclidean distance

- store the vector $\vec{pq}$
- squared Euclidean distance

Pros and cons

distance	exact	isotropic	storage	DT
path-based distances	✗	✗	✓	✓
vector	✓	✓	✗	✓
squared Euclidean	✓	✓	✗	✓

Distance transform with path-based distances

First idea

weighted graph representation + shortest path algorithm (Dikjstra for instance)

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Second idea

Decompose the mask into several sub-masks + raster scan for each sub-mask $\Rightarrow$ propagate the min values

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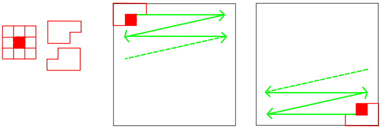
Example with 3×3 chamfer distance [Rosenfeld 66, Montanari 68]

$$DT(i,j) = \min_{(k,l) \in mask} (DT(i+k, j+l) + weight(k,l))$$



Initialisation :

$$DT(i,j) = 0 \quad si \quad (i,j) \notin S$$
$$DT(i,j) = +\infty \quad si \quad (i,j) \in S$$

Complexity : $\mathcal{O}(m.N^n)$

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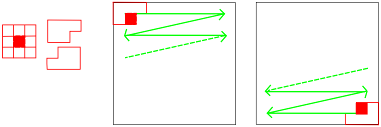

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Note : separable algorithm in $\mathcal{O}(\log^2 m.N^2)$ in 2D [Coeurjolly 2014]

Distance transform with Squared Euclidean distance

The problem

Let $p = (i, j) \in S \subset \mathbb{Z}^2$. We have :

$$DT(p) = \min_{q \notin S} \{d_2^2(p, q)\}$$

$$\Leftrightarrow$$

$$DT(i, j) = \min_{q(k,l) \notin S} \{(k - i)^2 + (l - j)^2\}$$

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Let's decompose [Saito et al. 94]

We can rewrite :

$$opt_x(i, {\color{red}j}) = \min_{q(k,{\color{red}j}) \notin S} \{(k - i)^2\}$$

and then

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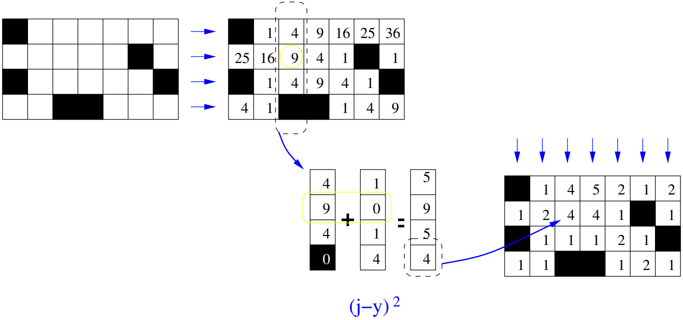
$\Rightarrow$ *paradigm = separable algorithm*

Distance transform with d_2^2 - example

Digital set S in white.



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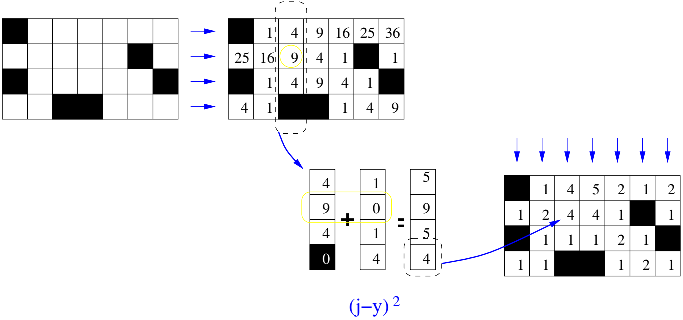
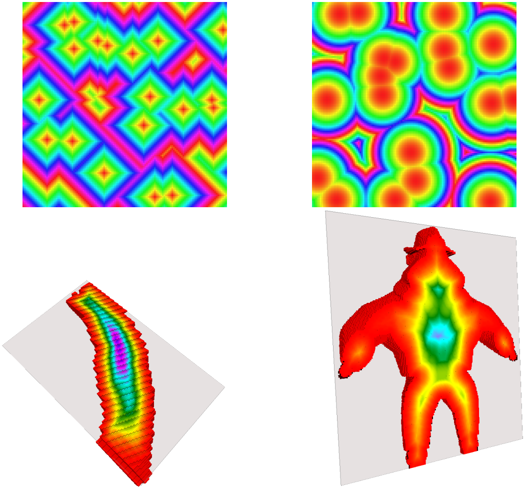


Complexity : $O(N^2)$ for a 2D $N \times N$ domain $\Rightarrow O(N^d)$ for a nD N^n domain.

Note : works also for any distance deriving from a L_p-norm (in particular d_1, d_∞).

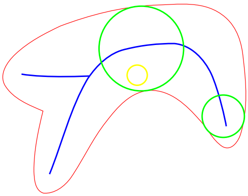
Images...



[Coeurjolly et al. 07] [DGtal Library]

Plan

Several definitions



Seminal definition [Blum 67]

Meeting points of a grassfire initialized on the shape boundary

Modern definitions

- set of centers of maximal balls
- set of points having at least two closest points on the boundary

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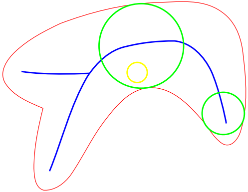

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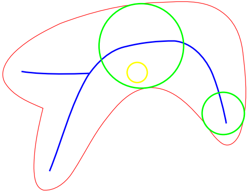

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For a digital set/object $S \subset \mathbb{Z}^2$

Medial Axis : set of centers of maximal balls inside S

Skeleton : digital object, topologically equivalent to S and *minimal*

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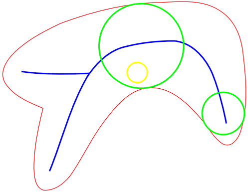

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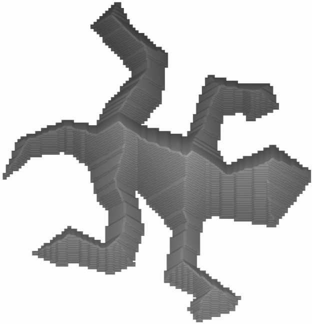
Medial axis and Distance Transform

Link

For a digital set S, and $p \in S$, $DT(p) =$ radius of the largest ball centered on p and included in S.

$\Rightarrow$ Medial Axis of S = local maxima of the DT



[K. Palágyi, http ://www.inf.u-szeged.hu/ palagyi/skel/skel.html]

Algorithm depends on the distance...

Path-based distance

Idea : to check whether a point $p \in S$ belongs to the Medial Axis, it is enough to compare with *a few neighbours*
$\Rightarrow$ Use a hash table

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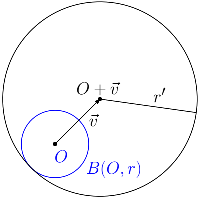
Example with the Chamfer distance



$$LUT(\vec{v}, r) = \min \{r' | B(O, r) \subseteq B(O + \vec{v}, r')\}$$

$$p \in MA \iff DT(p+\vec{v}) < LUT(\vec{v}, DT(p)) , \quad \forall \vec{v} \in \mathcal{V}.$$

Neighbourhood $\mathcal{V}$? : for d_∞, $\mathcal{V} = \{(\pm 1, \pm 1), (0, \pm 1), (\pm 1, 0)\}$.

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Squared Euclidean distance

Cannot pre-define a (limited) set of neighbours and radii
$\Rightarrow$ New tool : power diagram

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Rewriting S

$$
\begin{aligned}
\textit{Lapalissade}: \quad S &= \bigcup_{p \in S} B(p, DT(p)) \\
&= \{(i,j) \mid \exists p,\ (i-x_p)^2 + (j-y_p)^2 < DT(p)\} \\
&= \{(i,j) \mid \exists p,\ \underbrace{DT(p) - (i-x_p)^2 - (j-y_p)^2 > 0}_{\text{paraboloid}}\}
\end{aligned}
$$

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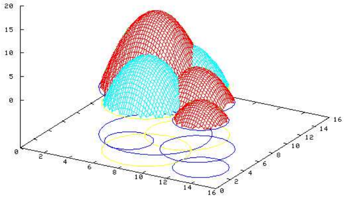

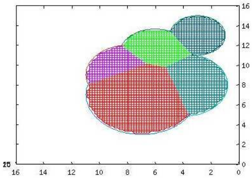

[Coeurjolly et al. 07]

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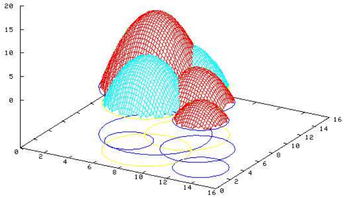

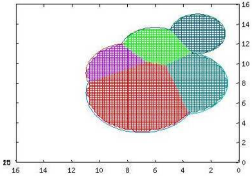

[Coeurjolly et al. 07]

⇒ *Highest paraboloïds* are enough to define S : others correspond to non-maximal balls

⇒ algorithmic tool from computational geometry = *Power Diagram*

Medial Axis and Digital Set reconstruction

Property

S is exactly defined by the set of centers of the medial axis + radii

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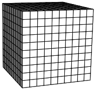
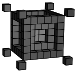
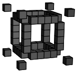
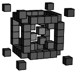
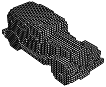
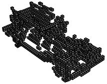
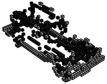
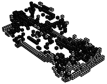
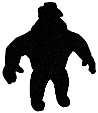
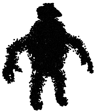
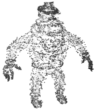
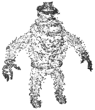
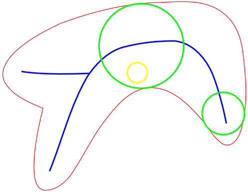
$\Rightarrow$ simplification algorithms [Ragnemalm et al. 91, Coeurjolly et al. 08]

Medial Axis and Digital Set reconstruction

Objet	$\mathcal{F} = AM(\mathcal{S})$	$\hat{\mathcal{F}}$ Ragnemalm et al.	$\hat{\mathcal{F}}$ Greedy
	 104	 56 (-46%) [<0.01s]	 66 (-36%) [< 0.01s]
	 1292	 795 (-38%) [0.1s]	 820 (-36%) [0.19s]
	 17238	 6177 (-64%) [48.53s]	 6553 (-62%) [57.79s]

Plan

Several definitions



Seminal definition [Blum 67]

Meeting points of a grassfire initialized on the shape boundary

Modern definitions

- set of centers of maximal balls
- set of points having at least two closest points on the boundary

$\Rightarrow$ For a *shape* $F \subset \mathbb{R}^2$, these definitions lead to a *1-dimensionnal topological centered equivalent of* F

For a digital set/object $S \subset \mathbb{Z}^2$

Medial Axis : set of centers of maximal balls inside S

Skeleton : digital object, topologically equivalent to S and *minimal*

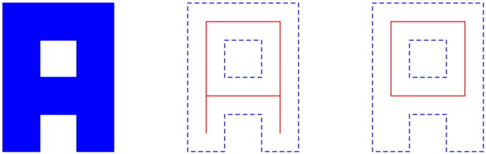
Skeleton

Let O be *a digital object* = digital set + topology (adjacency relations).

Principle

Withdraw some points of O (one by one) without "modifying the topology" :

- homotopy equivalence
- or homeomorphism ?



[Courtesy of C. Ronse]

Homotopy equivalence - Simple points

Simple point

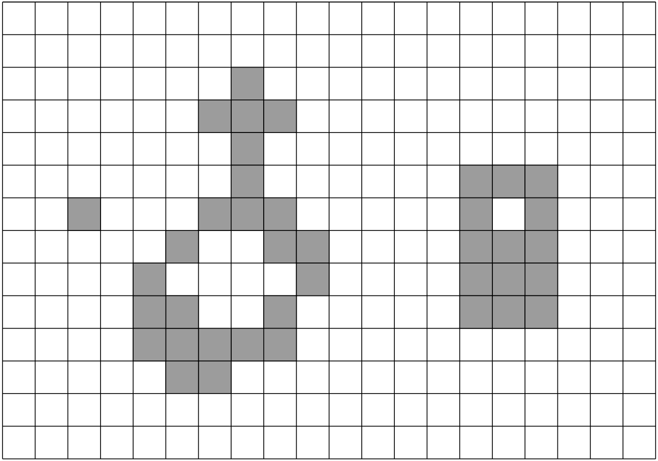
A point $p \in O$ is *simple* iff $O - \{p\}$ is homotopy equivalent to O.

In practice - Constraints

- a connected component of O cannot be deleted
- a connected component of O cannot be disconnected
- no connected component of O^c can be created
- no connected components of O^c can be merged

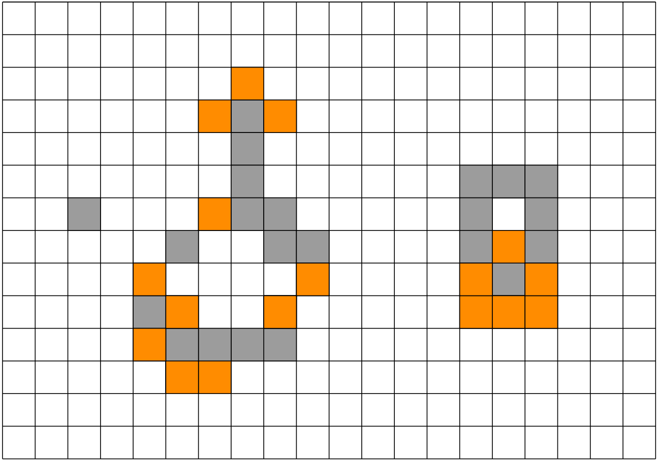
[Rosenfeld 70, Ronse 86, ...]

Example of simple points



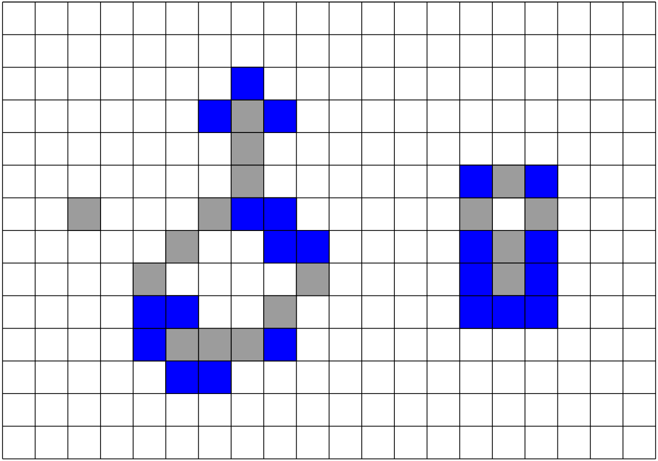
Example of simple points

Digital Object O is 4-connected.



Example of simple points

Digital Object O is 8-connected.

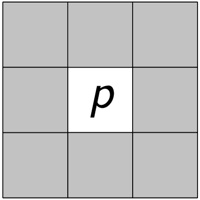


Characterization of simple points

From the definition, the characterization is *global*
$\Rightarrow$ at least $\mathcal{O}(n)$ to test *one* point in a domain of n points.

(Very) Local criterion

Let's count the connected components in the neighbourhood of p.



Neighbourhood of p

$\Rightarrow$ characterization in $\mathcal{O}(1)$!

[2D : Rosenfeld 79] [3D : Morgenthaler 81, Bertrand 94, Saha et al. 94,...] [4D : Kong 97, Couprie & Bertrand 09]

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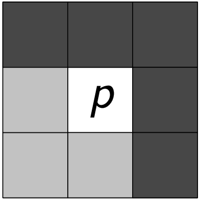

1 connected component for O
1 for O^c
$\Rightarrow$ p is simple

$\Rightarrow$ characterization in $\mathcal{O}(1)$!

[2D : Rosenfeld 79] [3D : Morgenthaler 81, Bertrand 94, Saha et al. 94,...] [4D : Kong 97, Couprie & Bertrand 09]

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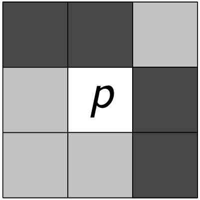

O is 4-connected

2 connected component of O 4-connected to p
2 connected component of O^c 8-connected to p
$\Rightarrow$ p is not simple

$\Rightarrow$ characterization in $\mathcal{O}(1)$!

[2D : Rosenfeld 79] [3D : Morgenthaler 81, Bertrand 94, Saha et al. 94,...] [4D : Kong 97, Couprie & Bertrand 09]

Characterization of simple points

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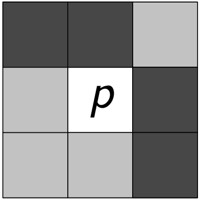

O is 8-connected

1 connected component of O 8-connected to p
1 connected component of O^c 4-connected to p
$\Rightarrow$ p is simple

$\Rightarrow$ characterization in $\mathcal{O}(1)$!

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Thinning algorithm

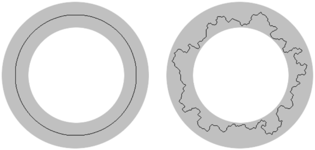
Simple sequential removal

While there exist p simple in O

- $O \leftarrow O \backslash p$

Order matters

How to choose p ?



[Chaussard 10]

Breadth-first thinning algorithm

Data: Digital Object O
Result: Digital Object $Sk(O)$
$Sk(O) \leftarrow O$
$Queue$: $SP \leftarrow \{p \in O \mid p \text{ is simple for } O\}$

Breadth-first thinning algorithm

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$Sk(O) \leftarrow O$
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    forall the $q \in Sk(O)$, q *neighbour of* p **do**
      $Next \leftarrow Next \cup \{q\}$
    end
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        forall the $q \in Sk(O)$, *q neighbour of p* **do**
            $Next \leftarrow Next \cup \{q\}$
        end
    end
    $SP \leftarrow \emptyset$
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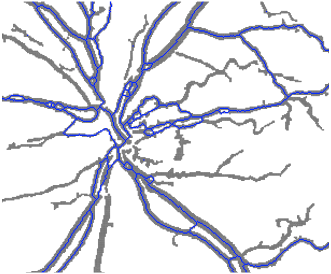
            $SP \leftarrow SP \cup \{p\}$
    end
end

Note : A priority function can also be used (for instance the Distance Transform).

Preserving the shape

Homotopy equivalence $\Rightarrow$ any digital object O with a single connected component reduces to a single point !



Preserving the shape

Homotopy equivalence $\Rightarrow$ any digital object O with a single connected component reduces to a single point !

Anchor points

Predicate defining un-removable points, for instance : points with one neighbour, points of the medial axis

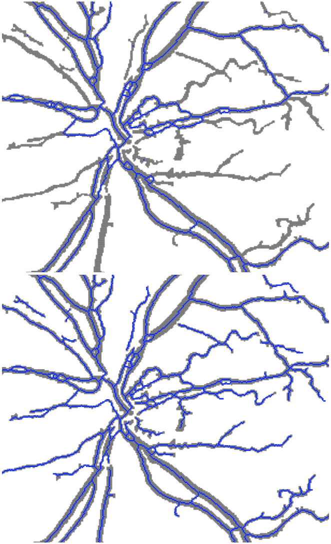
...

if *p is simple for* $Sk(O)$ *and* *not* $Anchor(p)$
then

$$Sk(O) \leftarrow Sk(O) \setminus \{p\}$$

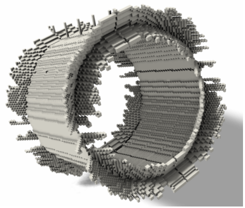
...

+ huge literature on the subject (see PhD thesis of [Chaussard 10] for instance)



Cellular grid space framework

In higher dimension, the skeleton does not always have nice properties (remaining 3d parts for instance).



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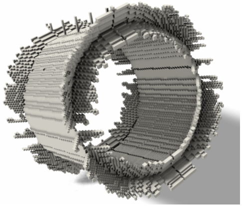
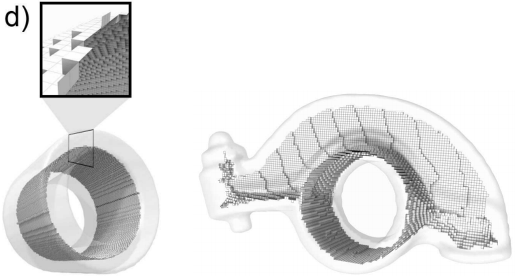
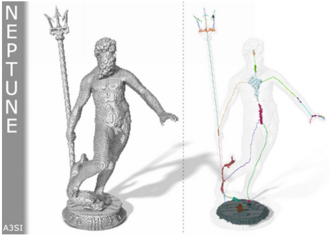


⇒ Use the cellular grid space : need to define simple cells, etc.





[Image from Chaussard 10] [see also Mazo 11]

Outline

Transformations

Measurements

- Multigrid convergence
- Area estimation
- Tangent, normal, length estimation
- Curvature and higher derivatives estimation

Goal : compute geometric quantities

Consider a family $\mathcal{F}$ of shapes in $\mathbb{R}^n$ (fulfilling given properties).

Global geometric quantities

For a given shape $S \in \mathcal{F}$, compute :

- its area (volume)
- its perimeter (area of its boundary)

Local geometric quantities

For any point $x \in \partial S$, $S \in \mathcal{F}$, compute :

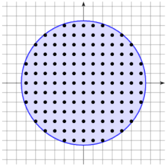
- its tangent, normal vector
- its curvature

Plan

Is the computation accurate, truthful ?

Consider :

- a grid $\mathbb{G}$ of resolution $1/h$, $h > 0$ (size of the pixels).
- a digitization process Dig_h such that $\mathrm{Dig}_h(S) = S \cap (h\mathbb{Z}^2)$ is the digitized version of $S \in \mathcal{F}$



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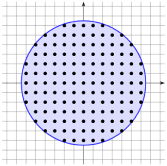
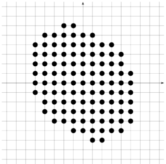
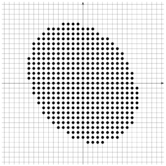


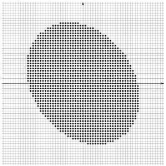
What happens when $h \to 0$?



$\mathrm{Dig}_h(S)$



$\mathrm{Dig}_{h/2}(S)$

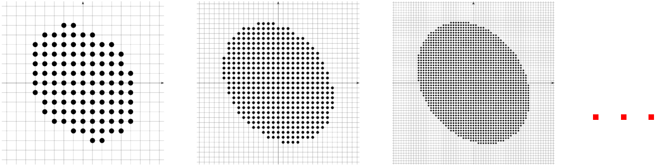


$\mathrm{Dig}_{h/4}(S)$

. . .

. . .

Global geometric estimators - multigrid convergence



$\mathrm{Dig}_h(S)$ $\mathrm{Dig}_{h/2}(S)$ $\mathrm{Dig}_{h/4}(S)$ ...

Multigrid convergence [Serra82]

Let $\mathcal{F}$ be a family of shapes. The geometric estimator $\hat{\epsilon}$ is said to be multigrid convergent for $\mathcal{F}$ toward the geometric descriptor ϵ iff $\forall S \in \mathcal{F}$

$$\lim_{h \to 0} |\hat{\epsilon}(\mathrm{Dig}_h(S)) - \epsilon(S)| \leq \tau_S(h)$$

$$\lim_{h \to 0} \tau_S(h) = 0$$

The *speed of convergence* is given by $\tau_S(h)$.

Local geometric estimators - multigrid convergence

Local multigrid convergence

Let $\mathcal{F}$ be a family of shapes. The geometric estimator $\hat{\epsilon}$ is said to be multigrid convergent for $\mathcal{F}$ toward the geometric descriptor ϵ iff $\forall S \in F, \forall x \in \partial S$,

$$\forall y \in \partial \mathrm{Dig}_h(S) \text{ with } \|y - x\|_1 \leq h,$$

$$|\hat{\epsilon}(\mathrm{Dig}_h(S), y) - \epsilon(S, x)| \leq \tau_{S,x}(h)$$

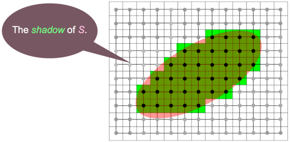
$$\lim_{h \to 0} \tau_{S,x}(h) = 0$$

The *speed of convergence* is given by $\tau_{S,x}(h)$.

[See Coeurjolly et al. 12 for a survey on digital estimators]

Plan

First example : area estimation



Consider the *shadow* digitization scheme Dig_h.

For a shape $S \in \mathcal{F}$, let's count the number of points in $\mathrm{Dig}_h(S)$, and define $\hat{\epsilon}(\mathrm{Dig}_h(S)) = h^2.|S \cap (h\mathbb{Z}^2)|$.

- For the family of convex shapes $\mathcal{F}$, $\tau_X(h) = O(h)$ [Gauss,Dirichlet]
- For the family of $\mathcal{C}^3$-convex shapes $\mathcal{F}$, $\tau_X(h) = O(h^{\frac{15}{11}+\epsilon})$ [Huxley 90]

Speed of convergence

h	1	0.1	0.01	0.001	...
$h^{\frac{15}{11}}$	1	0.04328	0.00187	0.00008	...

Plan

Transformations

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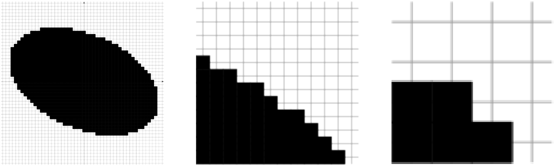
Length estimation - first trial

Let C be a digital curve, defined by a sequence of elementary displacements.

Question

Can we define a *convergent length estimator* of C by counting the number of elementary displacements ?



Remark

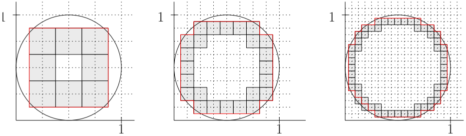
Even if the Hausdorff distance between the digital boundary and the shape boundary ∂S tends towards 0 when $h \to 0$, "stairs effect" remains.

Local weights - Stair effect

Let's try for different digitization schemes.

"Shadow" of a shape

Let S be a disk of radius $\frac{1}{2}$, and Dig_h be the "shadow".
Let C be $\partial\mathrm{Dig}_h(S)$.



Length of C tends to 4 instead of π !

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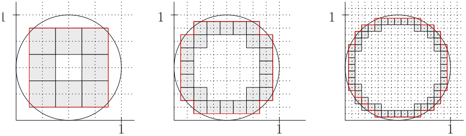
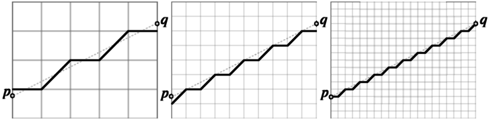


Length of C tends to 4 instead of π !

Digitization of a segment

Let S be a segment of slope $\alpha \in [0,1]$ and length l.
$\mathrm{Dig}_h(S)$ is the digitization that "chooses the closest point".



Length of S is equal to $\frac{5(1+\sqrt{(2)})}{2}$ $\forall n \geq 1$ instead of $\frac{5\sqrt{5}}{2}$!

[Klette, 2004]

Doomed local weights

Local weights estimator

- Consider the decomposition of C into parts of m elementary displacements such that $C = w_1 w_2 \dots w_n \lambda$
- Give a weight $p(.)$ to each w_i
- Define the estimator $\hat{\epsilon}(C) = \sum_{i=1}^{n} p(w_i)$

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Result when $C = \mathrm{Dig}_h(S)$, S a straight segment

For all m and all $p(.)$, the set of segments for which the estimator converges to the length of S is countable. Meaning that most of the time, the estimator does not converge. [Tajine, Daurat 03].

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Another solution ?

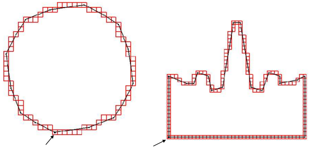
Decompose C into parts such that the "length" of the parts adapts itself to the curve. $\Rightarrow$ use digital segments

[see also semi-local estimators in Daurat et al. 11, Mazo & Baudrier 14]

Length estimation through polygonalization

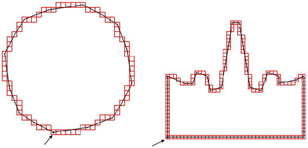
Principle

Compute a polygon from the digital curve C using digital straight segments.
$\Rightarrow$ length of C = length of the polygon.



Length estimation through polygonalization

Principle



Compute a polygon from the digital curve C using digital straight segments.
$\Rightarrow$ length of C = length of the polygon.

Several approaches

- greedy as long as possible DSS [Kovalesvsky et al. 92]
- Miminum Length Polygon : shortest curve that separates inside from outside [Sloboda 98, Provençal & Lachaud 09]
- Faithful Polygon : polygon that preserves convex and concave parts [Roussillon & Sivignon 11]

Multigrid convergence (theoretical and/or experimental) in $\mathcal{O}(h)$.

Tangent estimation

Let $C = p_i$ be a digital curve in $\mathbb{Z}^2$.
Goal : estimate the first derivative for all $p \in C$.

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Maximal Digital Segment

A digital segment S_{ij} is *maximal* on C if there is no segment $S \subset C$ such that $S_{ij} \subset S$.

Tangent estimation

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Maximal Digital Segment

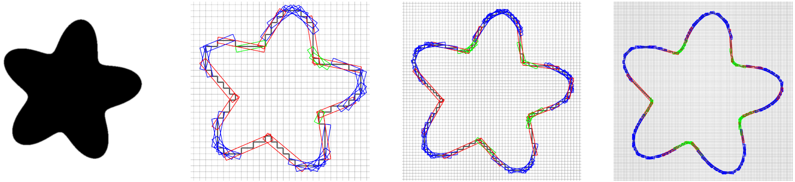
A digital segment S_{ij} is *maximal* on C if there is no segment $S \subset C$ such that $S_{ij} \subset S$.

Algorithmically

- Maximality can be tested locally, checking $S_{i-1,j}$ and $S_{i,j+1}$
- algorithms to add or remove a point at the front or at the back of a DSS in $\mathcal{O}(1)$ [Debled & Reveilles 95, Lachaud et al. 07]

$\Rightarrow$ algorithm in $\mathcal{O}(|C|)$ to compute all the maximal DSS on C.

Maximal Digital Segments



S $\mathrm{Dig}_h(S)$ $\mathrm{Dig}_{\frac{h}{2}}(S)$ $\mathrm{Dig}_{\frac{h}{4}}(S)$

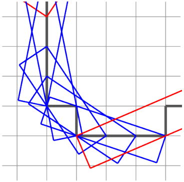
Swiss knife

- convergent tangent estimator
- convergent length estimator
- convex, concave parts, extremal points

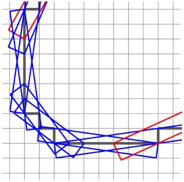
[Courtesy of T. Roussillon, DGtal]

Maximal Digital Segments

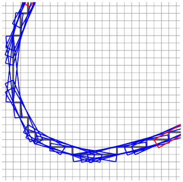
S



$\mathrm{Dig}_h(S)$



$\mathrm{Dig}_{\frac{h}{2}}(S)$



$\mathrm{Dig}_{\frac{h}{4}}(S)$

Swiss knife

- convergent tangent estimator
- convergent length estimator
- convex, concave parts, extremal points

[Courtesy of T. Roussillon, DGtal]

Maximal Digital Segments

Theorem [Lachaud et al. 07]

Let $x \in \partial S$. The direction of any maximal digital segment of $\mathrm{Dig}_h(S)$ that covers x converges to the tangent at x when $h \to 0$.

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Convergent length estimator

- "Digitize" $\int_0^1 t(s)ds$: $\widehat{Length}(\mathrm{Dig}_h(S)) = \sum_{e \in \partial \mathrm{Dig}_h(S)} \hat{t}(e).t_{elem}(e)$.
- Convergence speed : $\mathcal{O}(h^{\frac{1}{3}})$ ($\mathcal{O}(h^{\frac{4}{3}})$ experimentally)

[Coeurjolly & Klette 04, Lachaud 06]

Plan

Transformations

Measurements

- Multigrid convergence
- Area estimation
- Tangent, normal, length estimation
- Curvature and higher derivatives estimation

Curvature

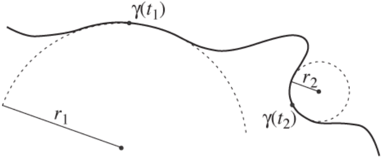
Let $\gamma(s)$ be a (at least $\mathcal{C}^2$) curve. The curvature κ along γ is given by :

Definitions

(i) norm of the second derivative $\kappa(s) = |\frac{d^2\gamma}{ds^2}|$

(ii) derivative of the tangent orientation $\kappa(s) = \frac{d\phi}{ds}$

(iii) inverse of the osculating circle radius $\kappa(s) = \frac{1}{r(s)}$



Families of digital curvature estimators (1/2)

Many algorithms that *mimic* these definitions :

- in (i) (ii) convolutions (local weighted means) are used to mimic derivatives [Worring & Smeulders 93, Feschet & Tougne 99, ...]

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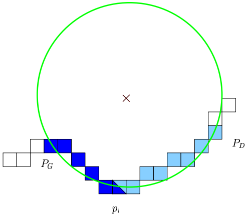

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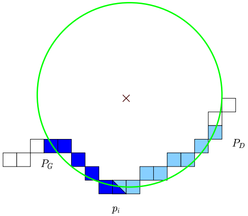

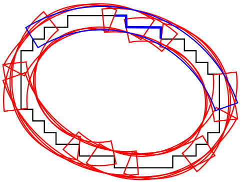

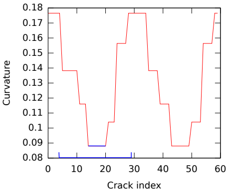

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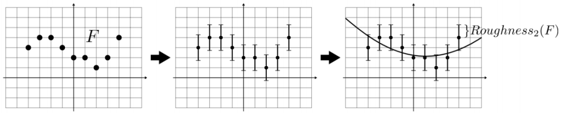

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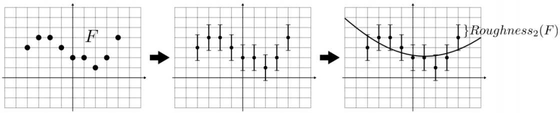
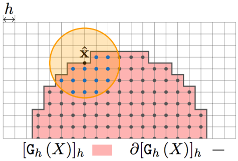
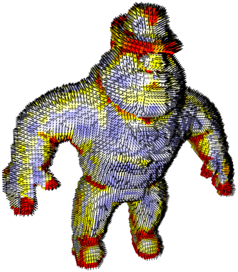
  - not parameter-free
- integral invariant estimator [Pottman et al. 07, Coeurjolly et al. 14, Levallois 15]
  - multigrid convergent in $\mathcal{O}(h^{\frac{1}{3}})$
  - can be parameter-free

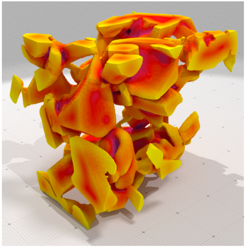




For 3D Digital Objects

- Normal vector estimation : multigrid convergent in $\mathcal{O}(h^{\frac{1}{8}})$ and stable algorithm [Cuel et al. 14, ...]
- Surface area : as for length estimation, integrate normals
- Mean and Gaussian curvature : integral invariant estimators are multigrid convergent in $\mathcal{O}(h^{\frac{1}{3}})$ [Coeurjolly et al. 14, Levallois 15]





The last words...

A short and incomplete overview

For instance, what about :

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Books to go further

- *Géométrie discrète et images numériques*, 2007. Collective book (in french).
- *Digital Geometry*, 2004. R. Klette & A. Rosenfeld (in english)

DGtal

Digital Geometry Tools and Algorithms

- aim at gathering digital geometry algorithms in a common programming framework
- open-source, collaborative library
- Let's try it during lab session this afternoon !

Thank you !